5/24/24 **Infinitesimal Structure of Bun_G** (Math-Physics seminar at Perimeter Institute)

Let k = alg. closed field (of char. 0 or p)

X = smooth projective curve $/k$.

G = semisimple group $/k$, $\mathfrak{g} := Lie(G)$

$Bun_G X$ = Moduli stack of principal G-bundles $/X$.

$\mathcal{M} = \{ P \in Bun_G(X)(k) \mid H^0(X, \mathfrak{g}_P) = 0 \}$, $\mathfrak{g}_P := \mathfrak{g} \times P / G$, the associated bundle.

= Moduli space of principal G-bundles over X, with finite group of automorphisms (so is the smooth locus of the moduli stack $Bun_G X$)

Goal: Give several concrete realizations of the infinitesimal jet (= higher order cotangent space) spaces of $\mathcal{M}$.

↳ Verbally "n^{th} order Taylor expansion of functions on $\mathcal{M}$"

- This is accomplished by giving two canonical pairings with conformal blocks for Affine Kac-Moody Lie algebras!

§1 1st Order Jets

- Fix $x \in X$ (Later, we'll explain the independence of choosing a base point!)

- Let $\mathcal{O}_x := \hat{\mathcal{O}}_{X,x} \simeq k[[t_x]] \rightsquigarrow D_x = \operatorname{Spec} \mathcal{O}_x$ = "formal disk of $x \in X$"

$K_x := \operatorname{Frac} \mathcal{O}_x \simeq k((t_x)) \rightsquigarrow D_x^\circ = \operatorname{Spec} K_x$ = "formal punctured disk of $x \in X$".

$\mathcal{O}_{out} := \mathcal{O}_X |_{X \setminus x} \rightsquigarrow$ punctured curve.

Thm [Beauville-Laszlo] There exists a 1-point uniformization

$$\pi : G(\mathcal{O}_{out}) \backslash G(K_x) / G(\mathcal{O}_x) \xrightarrow{\sim} Bun_G(X)$$

isomorphism of stacks.

Comment: Isom given by trivializing bundle over $X \setminus x$ & D_x (both affine) & then giving a transition map over the intersection D_x°. This then allows us to work locally, over formal power series.

As a consequence, we find:

$$T_P \mathrm{Bun}_G(X) = \mathfrak{g}_{K_x} / (\varphi \mathfrak{g}_{O_x} \varphi^{-1} + \mathfrak{g}_{out})$$

where $\mathfrak{g}_{K_x} \simeq \mathfrak{g} \otimes k((t))$, $\mathfrak{g}_{O_x} \simeq \mathfrak{g} \otimes k[[t]]$, $\mathfrak{g}_{out} \simeq \mathfrak{g} \otimes \mathcal{O}(X \setminus x)$, $G(K) \xrightarrow{\pi} \mathrm{Bun}_G$, $\varphi \longmapsto P$

Using a Cech cover $\{X \setminus x, D_x\}$, can also show

$$\boxed{T_P(\mathrm{Bun}_G X) = \mathfrak{g}_{K_x} / (\varphi \mathfrak{g}_{O_x} \varphi^{-1} + \mathfrak{g}_{out}) \underset{\text{Cech}}{\simeq} H^1(X, \mathfrak{g}_P)} \underset{\text{Serre}}{\simeq} H^0(X, \mathfrak{g}_P^* \otimes \Omega_X)^*.$$

So dually, this provides 1^{st} order approximation of functions on $\mathrm{Bun}_G(X)$.

§ 3 Higher order jets.

Since we treat char 0 & p simultaneously, we must introduce a notion of "divided powers."

Suppose first $V = k^n$ (fin dim vector space)

Define

$$\mathrm{Sym}^{PD}(V) := \bigoplus_{i \geq 0} (V^{\otimes i})^{S_i} = \text{"free-PD polynomial algebra } /k\text{"}$$

$$\mathrm{Sym}(V) := \bigoplus_{i \geq 0} V^{\otimes i} / \langle v \otimes w = w \otimes v \rangle$$

isomorphic if char $k = 0$.

$$\boxed{t^{(i)} = \frac{t^i}{i!}}$$

So, if $n = 1$, $\mathrm{Sym}^{PD}(k) = k[t^{(i)} : i \geq 0] / \left(t^{(n)} \cdot t^{(m)} = \binom{n+m}{n} t^{(n+m)} \right)$

Remark: there's additional "divided power" structure on $\mathrm{Sym}^{PD}(k)$ which says you may do: $(t^{(n)})^{(m)} = \frac{(mn)!}{n!(m!)^n} t^{(nm)}$

Lem There's a canonical perfect pairing

$$\langle , \rangle : \mathrm{Sym}^{PD}(V) \times \mathrm{Sym}(V^*) \longrightarrow k.$$

Upshot: In char. p, perfect pairings come when one side attaches denominators & the other does not.

Now, sheafify: let $\mathcal{M}$ = smooth scheme/k.

Let $\mathcal{O}^e := \mathcal{O}_\mathcal{M} \otimes \mathcal{O}_\mathcal{M}$, $I = \operatorname{Ker}[\mathcal{O}^e \xrightarrow{\text{mult}} \mathcal{O}_\mathcal{M}] = \langle f\otimes 1 - 1\otimes f \rangle_f$

Def (1) $J^n(\mathcal{M}) := \mathcal{O}^e / I^{n+1}$ = sheaf of n-jets of functions on $\mathcal{M}$

"divided power envelope"

(2) $J^{n,PD}(\mathcal{M}) := \Gamma_I(\mathcal{O}^e)/I^{n+1}$ = n^{th} PD-neighborhood of $\Delta: \mathcal{M} \hookrightarrow \mathcal{M}\times\mathcal{M}$

Due to Berthelot-Ogus.

Affine locally,

$\Gamma_I(\mathcal{O}^e)$ = free PD-polynomial algebra over $\mathcal{O}_\mathcal{M}$.

I^{n+1} = ideal gen'd by symbols of degree $n+1$.

(Again, (1) $\cong$ (2) if char $k = 0$).

The fiber of $J^{n,PD}(\mathcal{M})$ over $P \in \mathcal{M}$ is called the vector space of n^{th} order infinitesimal divided-power jet spaces of $P\in\mathcal{M}$.

Def $D^{crys}(\mathcal{M})_n := \mathcal{H}om_\mathcal{O}\left(J^{n,PD}(\mathcal{M}), \mathcal{O}\right)$ = sheaf of crystalline diff ops

$\Longrightarrow \langle , \rangle : D_P^{crys}(\mathcal{M})_n \times J_P^{n,PD}(\mathcal{M}) \longrightarrow k$ is perfect pairing by definition

By comparing basis,

$$D_P^{crys}(\mathcal{M}) \xrightarrow{\sim} U(T_P\mathcal{M}) \quad \text{isom of associative algebras.}$$

Recall, we computed $T_P\mathcal{M}$, and as a corollary,

$$U(T_P\mathcal{M}) \simeq \underbrace{U(\mathfrak{g}_k) \big/ \left((\mathrm{Ad}_\varphi \mathfrak{g}_0) U\mathfrak{g}_k + U\mathfrak{g}_k \cdot \mathfrak{g}_{out}\right)}$$

Denote by $\mathbb{M}_\varphi^{out}$ = "space of coinvariants."

Have induced PBW filtration, & as a corollary:

Cor There is a natural perfect pairing

$$\varphi : \mathbb{M}^{out}_{\varphi, n} \times J_p^{D,PD}(\mathcal{M}) \longrightarrow k.$$

Remark: Let $\varphi \in G(K)$. Then "Vacuum module w/ central charge 0"

is $\mathbb{M}_\varphi := U g_K / (\mathrm{Ad}_\varphi g_0) U g_K \left(\simeq \mathrm{Ind}_{g_0}^{g_K} \mathbb{1}_\varphi\right)$

& Corollary may be interpreted as saying infinitesimal jets of Bun_G are precisely the conformal blocks associated to affine Kac-Moody Lie alg. w/ central charge 0.

§ 3 Log Differential Forms

We provide another description of $J_p^{D,PD}(\mathcal{M})$ which illustrates the role of configuration space:

Recall the Fulton–Macpherson compactification:

$$p : \hat{X}^n \longrightarrow X^n \quad \text{w/ properties:}$$

(1) $p^{-1}(D) = \bigcup_{|I| \geq 2} \hat{D}_I$ is a normal crossing divisor, where $D = \bigcup \{x_i = x_j\}$.

(2) $\hat{D}_I \simeq \hat{\mathbb{P}}^I \times \hat{X}^{[n]/I}$, $[n]/I = [n] \setminus I \cup \{I\}$.

(3) $\hat{X}^n = \coprod_{T = [n]\text{-tree}} S_T$ ← strata, $\overline{S_T} \simeq \hat{\mathbb{P}}_T \times \hat{X}^J$, $J = \#$ connected components

↖ Explain why trees! $\hat{\mathbb{P}}_T = \hat{\mathbb{P}}^{I_1} \times \dots \times \hat{\mathbb{P}}^{I_s}$

Remark (3) $\Rightarrow \{\hat{X}^n\}_{n \geq 1}$ is in fact a "module" over the topological operad $\{\hat{\mathbb{P}}^n\}_{n \geq 1}$

Def $\cdot\ \Omega_{\hat{X}^n, \hat{X}^n}$ denote the sheaf of $\overset{\underline{\text{Top degree!}}}{\vee}$ differential forms on $\hat{X}^n$,

regular on $\mathring{X}^n$, but with simple logarithmic poles along
$\hat{D} = \hat{X}^n \setminus \mathring{X}^n$.

• $\Omega(\hat{X}^n, \mathring{X}^n) := \Gamma(\hat{X}^n, \Omega_{\hat{X}^n, \mathring{X}^n})$.

<u>Ex</u> $\Omega(\hat{\mathbb{P}}^{[3]}, \mathring{\mathbb{P}}^{[3]}) = \{\lambda_{23}\, d\log(z_{23}) + \lambda_{31}\, d\log(z_{31}) + \lambda_{12}\, d\log(z_{12}) : \lambda_{ij} \in k,\ \lambda_{12}+\lambda_{23}+\lambda_{31}=0,\ z_{ij} = z_i - z_j\}$

$\| \; k^2$

<u>Lem</u> $\mathrm{Res} : \mathrm{Lie}(I) \times \Omega(\hat{\mathbb{P}}^I, \mathring{\mathbb{P}}^I) \longrightarrow k$

is a perfect pairing

Let $\varphi_I^* : \mathfrak{g}^* \longrightarrow \mathfrak{g}^{*\otimes I} \otimes \mathrm{Lie}(I)$, $I = [n]$

$A \longmapsto \left(x_1 \otimes \cdots \otimes x_n \longmapsto A([x_{\sigma(1)},[\dots,[x_{\sigma(n-1)}, x_n]]])\right)_{\sigma \in S_{n-1}}$

We may upgrade φ_I to an $\mathcal{O}_{\hat{D}_I}$-module hom:

$$\begin{tikzcd} \hat{D}_I \arrow[d, "p_2"] \\ \hat{X}^{[n]/I} \end{tikzcd}$$

$$\begin{tikzcd} \psi_I^* : p_2^*\left(\hat{\mathfrak{g}}_p^{*\boxtimes [n]/I} \otimes \Omega_{\hat{X}^{[n]/I}, \mathring{X}^{[n]/I}}\right) \arrow[r] & \hat{\mathfrak{g}}_p^{*\boxtimes n}\Big|_{\hat{D}_I} \otimes \Omega_{\hat{D}_I, \mathring{D}_I} \\ & \hat{\mathfrak{g}}_p^{*\boxtimes n} \otimes \Omega_{\hat{X}^n, \mathring{X}^n} \arrow[u, "\mathrm{Res}_{\hat{D}_I}"'] \end{tikzcd}$$

<u>Rmk</u> Fiberwise, $\psi_I^*\big|_{x \in \hat{D}_I} = \mathrm{Id} \times \varphi_I^*$.

<u>Remark</u>: φ_I^* injective ($\Rightarrow \psi_I^*$ embedding) because $\mathfrak{g}$ semisimple, &

$\mathrm{Im}(\psi_I^*)$ is a locally free sheaf on $\hat{D}_I$ w/ fiber $\mathfrak{g}^{*[n]/I}$.

<u>Main Def</u> Define the "BG sheaf" on $\hat{X}^n$ by

$$\hat{G}_n := \left\{\omega \in (\hat{\mathfrak{g}}_p^*)^{\boxtimes n} \otimes \Omega_{\hat{X}^n, \mathring{X}^n} \;\middle|\; \mathrm{Res}_{\hat{D}_I}(\omega) \in \mathrm{Im}\, \psi_I^* \ \forall\, |I| \geq 2\right\}$$

Mention: the constraint says the coefficients of log forms are built by nested cobracket expressions.

Main Thm ($G_\circ$)
$$\Gamma(\hat{X}, \hat{G}_n)^{-S_n} \simeq J_p^{n,PD}(M).$$

Pf strategy:

① We define an analogues "sheaf" G_n on X^n such that
$$\Gamma(\hat{X}^n, \hat{G}_n)^{-S_n} \overset{\text{thm}}{=} \Gamma(\hat{X}^n, p^* G_n)^{-S_n} \overset{\text{Def}}{=} \Gamma(X^n, G_n)^{-S_n}$$

② We use the proof of Beilinson-Drinfeld to show the canonical residue pairing
$$\mathrm{Res} : \mathbb{M}_n \times \Gamma(D_x^n, G_n)^{-S_n} \longrightarrow k$$
Spec $k[[t_1, \dots, t_n]]$.

is perfect.

Rmk: ② says Jets of $Gr_G = \Gamma(D_x^n, G_n)^{-S_n}$!

③ We show this pairing induces a perfect pairing
$$\mathrm{Res} : M_n^{out} \times \Gamma(X^n, G_n)^{-S_n} \longrightarrow k$$

Rmk: Say few words about proof, & dependence of $x \in X$.

Concluding remark: There's some sort of operadic structure

lurking:
$$\Omega^{PD}_{\hat{X}} \langle g_p^* \rangle := \bigoplus_{n \geq 1} \Gamma\left(\hat{X}^n, \hat{g}_p^{* \boxtimes n} \otimes \Omega_{\hat{X}^n, \hat{x}^n} \otimes \omega_{\hat{X}^n}\right)^{S_n}$$
det. line bundle

"=" free divided power comodule gen'd by g_p^*.

$$\bigoplus_{n\geq 1} \Gamma(\hat{X}^n, \hat{\mathcal{G}}_n \otimes \omega_{\hat{X}^n})^{S_n} \left(\overset{\text{thm}}{=\!=} \bigoplus \mathcal{J}_p^{n,PD}(\mathcal{M}) \right)$$

carries some operadic structure too.

E.g.,

$$\mathrm{Res}_{\hat{D}_I} : \hat{\mathcal{G}}_n \Big|_{\hat{D}_I} \longrightarrow \left(\hat{\mathcal{G}}_{[n]/I} \otimes \hat{g}_p^{*(I-1)} \right) \boxtimes \Omega_{\hat{\mathbb{P}}^I, \mathbb{P}^I}$$